Characterizing the process of reaching consensus for social systems

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A novel way of characterizing the process of reaching consensus for a social system is given. The foundation of the characterization is based on the theorem which states that the sufficient and necessary condition for a system to reach the state of consensus is the occurrence of *communicators*, defined as the units of the system that can directly communicate with all the others simultaneously. A model is proposed to illustrate the characterization explicitly. The existence of *communicators* provides an efficient way for unifying two systems that a state of consensus is guaranteed after the mergence.

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The appearance of system-wide harmonic behaviors, such as the globally coordinated movements for the units of a system, the consensus of opinions for a public event in a society, and etc., can be observed very often for different systems in different situations[1–5]. It is very remarkable that there is no center control in the first place for the occurrence of such global coordinations. Then, it should be interested to know the kinematic scenario for the arising of coordinated behaviors. In this Letter, we intend to give a novel way of characterizing the process of reaching the state of consensus for social systems. The cornerstone for the characterization is the identification of *communicators*, defined as the units of a system that can directly reach all the others at some instant in the time evolution of a system. Different units may start to act as *communicators* at different times, but, the units have remained to be in the same status once they become *communicators*. Then, we can classify the *communicators* into different levels according to their first appearance times. The *primary communicators* are referred to those appearing in the earliest, and they may correspond to the hub-units which are those with large values of degree in a social network. As the distribution of hub-units has a strong effect on the scaling behavior of the relaxation time towards the state of system-wide coordination from a strongly disorder one[6], we will show that the presence of *communicators* is the sufficient and necessary condition for a system to achieve the state of consensus. Thence, the process of reaching the state of consensus can be viewed as the sequential appearance of *communicators*. By employing the Watts-Strogatz networks for the social connections, we propose a simple model for the transition matrix to illustrate the sequential pattern explicitly. The model study indicates that the *communicators* may appear in different levels except the case of regular Watts-Strogatz network for which, all the *communicators* are *primary*, and the *communicators* with larger values for the degree of connection, in general, appear earlier. The characterization may provide useful applications to different situations. An example of applications, the mergence of two groups, is given. As two groups, which have the respective state of consensus, are merged together, our characterization can provide an efficient way of combination that guarantee the existence of a reachable state of consensus for the combined system.

Consider a system of N units with the communication paths specified by the connection edges. The distribution of edges is given by a $N \times N$ connection matrix Γ with the entries given as $\gamma_{ij} = 1$ for the existence of a directional edge from unit j to i, and $\gamma_{ij} = 0$ otherwise. An attitude-variable, denoted as x_i for the unit i with the value in the range $[0, 1]$, is assigned to an unit to represent its degree of favor towards an event; the attitude can be viewed as complete disagreement for the value 0, neutrality for $1/2$, and complete agreement for 1. As all variables take the same value, $x_i = c$ for $c \in [0, 1]$ and $i = 1, 2, \cdots, N$, the system is said to be in the state of consensus c. The time evolutions of the variables x_i in discrete time-steps are given as

$$X(t+1) = M \cdot X(t), \qquad (1)$$

where M is the transition matrix and $X(t) = (x_1(t), x_2(t), \cdots, x_N(t))^\tau$ with the superscript τ for the transpose. The off-diagonal entry m_{ij} of the matrix M gives the fractional rate of the influence from unit j to unit i, and the diagonal entries m_{ii} defines the fractional rate of persistence on the x_i value. The explicit form of M depends on the model, but we assume that the feedback from the self-persistence and the environmental influence is positive, this renders the matrix M to be non-negative..Furthermore, the fractional rates are normalized, $\sum_{j=1}^{N} m_{ij} = 1$ for $i = 1, 2, \cdots, N$, then, the x_i values always lie in the range $[0, 1]$ during the course of time evolution. These constraints for the matrix M make the transpose of M, M^τ, to be a stochastic matrix.

As M is a non-negative square matrix, the Perron-Frobenius theorem can be employed to assert the properties of the leading eigenvalue and the corresponding eigenvectors[8]. Here, we establish a theorem, which gives less restriction on the entries of M than the Perron-Frobenius theorem and provides the central theme for characterizing the process of reaching the state of consensus in a system. As the theorem followed by its proof

are given below, here we summarize the notation conventions and the corresponding definitions used in the theorem and the proof. (i) A bar is placed over the head of a vector, say $\overline{X}$, to denote a stochastic vector associated with the stochastic matrix M^{τ}, a stochastic vector is subject to the conditions that the components are positive and the sum of the components equals 1. (ii) Different types of norm for a vector are used for convenience: $\parallel X \parallel_{\infty}$is the super-norm of the vector X, defined as $\|X\|_{\infty} = \max\{|x_i|, 0 \le i \le N\}$; $\|X\|_1$ is the one-norm, defined as $\|X\|_1 = \sum_{i=1}^{N} |x_i|$; and $\|X\|_2$ is the two-norm, defined as $\|X\|_2 = \left(\sum_i x_i^2\right)^{1/2}$. Different types of norms are equivalent. (iii) The bracket of two vectors, $\langle X, Y\rangle = \sum_{i=1}^{N} x_i y_i$, denotes the inner product of X and Y.

Theorem: Suppose there exists an unit, say α, which can connect every other unit by a path of length n_0, that is, the entries of the α-th column of M^{n_0} are positive. Then, every trajectory of solution for Eq. (1) is leaded to a state of consensus,

$$X(k) \to cI \text{ as } k \to \infty. \tag{2}$$

Here, I denotes the column vector with each entry 1, and the c value, which signifies the state of consensus, is given as

$$c = \left\langle X(0), \overline{\Lambda}\right\rangle, \tag{3}$$

where $X(0)$ is the initial state of the system, and $\overline{\Lambda}$ is the eigenvector of M^{τ} with eigenvalue 1, $M^{\tau} \cdot \overline{\Lambda} = \overline{\Lambda}$. Here, the existence and the uniqueness of $\overline{\Lambda}$ are guaranteed by the Perron-Frobenius theorem[8]. For the speed of convergence, there exists $r \ge 1$ and $0 < \lambda < 1$ such that

$$\parallel X(k) - cI \parallel_{\infty} \le r\lambda^k \parallel X(0) - cI \parallel_{\infty}. \tag{4}$$

Moreover, the condition for leading to Eq. (2) is also necessary.

We first give the proof for the sufficient condition by showing the equivalent form of Eq. (4),

$$\|X(k) - cI\|_1 \le r\lambda^{[k/n_0]} \|X(0) - cI\|_1, \tag{5}$$

held for the system evolving to the time step k with $k \gg n_0$, where $[k/n_0]$ is the integer part of k/n_0. To show the inequality of Eq. (5), we consider the dynamics of the stochastic matrix M^{τ},

$$\overline{Y}(k) = M^{\tau} \cdot \overline{Y}(k-1). \tag{6}$$

As M^{n_0} has a positive column, the matrix $(M^{\tau})^{n_0}$ has a positive row. By defining

$$\tau = \sum_{i=1}^{N} \min\left\{[(M^{\tau})^{n_0}]_{ij}, 1 \le j \le N\right\}, \tag{7}$$

we have $0 < \tau < 1$, this yields $0 < \lambda < 1$ for $\lambda = 1 - \tau$. Following the theorem shown the Appendix of Ref. [7], we have

$$\left\|(M^{\tau})^k \left(\overline{Y}(0) - \overline{Z}(0)\right)\right\|_1 \le \lambda^{[k/n_0]} \left\|\overline{Y}(0) - \overline{Z}(0)\right\|_1, \tag{8}$$

for $k \gg n_0$, where $\overline{Y}(0)$ and $\overline{Z}(0)$ are two different initial states for the dynamics of Eq. (6). Furthermore, because of $M^{\tau} \cdot \overline{\Lambda} = \overline{\Lambda}$, we have

$$\left\langle M^{n_0} \cdot (X(0) - cI), \overline{\Lambda}\right\rangle = 0, \tag{9}$$

with c given by Eq. (3). Then, Eq. (5) is followed from Eqs. (8) and (9). To see this, we first notice that

$$\left\|M^k \cdot X(0) - cI\right\|_1 = \left\|M^k \cdot (X(0) - cI)\right\|_1. \tag{10}$$

Then, based on Eqs. (9) and (10) we have

$$\left\|M^k \cdot X(0) - cI\right\|_1 = \sum_{\alpha=1}^{N} \left|\left\langle X(0) - cI, (M^{\tau})^k \cdot \left(I_\alpha - \overline{\Lambda}\right)\right\rangle\right|, \tag{11}$$

where I_α is a column vector with only one non-zero entry of the value 1 locating at the α-th row, that is, $\sum_{\alpha=1}^{N} I_\alpha = I$. The Cauchy inequality is further applied to the right hand side of Eq. (11) to obtain

$$\|X(k) - cI\|_1 \le K_2^2 \|X(0) - cI\|_1 \left\{\sum_{\alpha=1}^{N} \left\|(M^{\tau})^k \cdot \left(I_\alpha - \overline{\Lambda}\right)\right\|_1\right\}, \tag{12}$$

where K_2 is the constant of equivalence between the two-norm and one-norm. Finally, we apply Eq. (8) to the second factor on the right hand side of (12) to obtain

$$\|X(k) - cI\|_1 \le 2 \cdot N \cdot K_2^2 \cdot \lambda^{[k/n_0]} \|X(0) - cI\|_1, \tag{13}$$

where we use the fact, $\left\|I_\alpha - \overline{\Lambda}\right\|_1 \le \|I_\alpha\|_1 + \left\|\overline{\Lambda}\right\|_1 = 2$.

For the proof of necessary condition, we set the initial state as $X(0) = \sum_{\alpha=1}^{N} I_\alpha$ and assume that the state of consensus is reached as

$$X(k) = M^k \cdot X(0) \to cI \tag{14}$$

for $k \to \infty$, where the c value is $c = \sum_{\alpha=1}^{N} c_\alpha$ with c_α corresponding to the value for the state of consensus when the initial state is I_α. By observing that the α-th column of M^k is $M^k \cdot I_\alpha$, we write

$$M^k = \left[M^k \cdot I_1, M^k \cdot I_2, \cdots, M^k \cdot I_N\right]. \tag{15}$$

Then, based on Eq. (14) we have

$$M^k \to [c_1 I, c_2 I, \cdots, c_N I] \tag{16}$$

for sufficiently large k. Suppose that all $c_\alpha = 0$ for $\alpha = 1, 2, \cdots, N$, this contradicts with the fact that $\rho(M) = \rho(M^{\tau}) = 1$, where $\rho(M)$ is the spectrum radius of M.

Thus, there is at least one $c_\alpha \neq 0$ in Eq. (16), and this gives the condition in the *theorem* as necessary.

Based on the *theorem*, we define the unit α as a *communicator* with the first appearance time n_0, if n_0 is the smallest one among all integers n that the α-th column is positive for the matrix M^n. Since the α-th column remains to be positive for M^n with $n > n_0$ if it is positive for M^{n_0}, an unit once become a *communicator*, it remains to be a *communicator* afterwards. Thus, we can classify the *commutators* into the *primary*, the *secondary*, and etc. in the order of the first appearance time from the earliest to the latest, and characterize the process of reaching a state of consensus by the sequential appearance of *communicators*. However, the *theorem* does not imply that all units have to become *commutators* before reaching the state of consensus, although this may occur for some forms of M.

For the purpose of illustration, we consider a simple model for the transition matrix M. The fractional rate for the persistence of the present attitude is assumed to be the same for all units and given by the parameter s with the setting $m_{ii} = s$, where $0 \leq s < 1$ and $i = 1, 2, \cdot \cdot \cdot, N$. For the off-diagonal entries, we assume that the environmental influence comes from the connected neighbors given by the connection matrix Γ. Moreover, the average of the attitudes of the neighbors is used to represent the social atmosphere faced by an unit. These amount to set the off-diagonal entries as $m_{ij} = (1 - s)\, \gamma_{ij}/z_i$, where γ_{ij} are the entries of Γ, and $z_i = \sum_{j=1}^{N} \gamma_{ij}$ is the inward degree of the unit i. The undirected Watts-Strogatz networks are used to define the connection matrices Γ. We first place N units around a circle with the degree of an unit k_0 connecting to the right and to the left neighbors symmetrically; then a value, called rewiring probability p, is assigned to rewire the edges randomly[9]. Consequently, the members of Watts-Strogatz networks have different degrees of randomness from regular lattices ($p = 0$) to random graphs ($p = 1$). For a symmetric Γ, the matrix M is symmetric and stochastic. Then, the eigenvector of the eigenvalue 1 for M^τ is $\overline{\Lambda} = (1/N)\, I$, this leads to the state of consensus as the mean value of the initial state, $c = \sum_{i=1}^{N} x_i\,(0)\,/N$, which gives the state of consensus $c = 1/2$ for a strongly disorder initial state. By setting set $N = 1000$, $k_0 = 4$, and $p = 0.1$ for the network and $s = 0.3$ for the self-persistence, we show the results in Fig. 1(a) for the first appearance time t_c of a *communicator* n in a trajectory from a strongly disorder state to the state of consensus $c = 1/2$ (the upper part) and the corresponding degree of connection k of the *communicator* n (the lower part). The results indicate that there does not exist a definite relation between the first appearance time of a *communicator* and its degree of connection. However, the corresponding k values, in general, are larger for the *communicators* that the t_c values are smaller as shown in Fig. 1(b), where, based on the results of Fig. 1(a), the average of the first appearance time of the *communicators* with the same k value, $\langle t_c \rangle$, as a function of k is shown. It is worthy to notice that contrary to the sequential appearance of *communicators* for the Watts-Strogatz networks with $p \neq 0$, all units appear to be *communicators* simultaneously for regular lattices ($p = 0$) owing to the indistinguishability between the units of the system.

[Figure Caption]Fig.1: (a) The first appearance time of a *communicator*, t_c (the upper part), and the corresponding degree of connection, k (the lower part), for different units of the system, n, where the unit n is labelled in accordance with the order of the t_c value from small to large. (b) The average value of the first appearance times of the *communicators*, $\langle t_c \rangle$, as a function of k for the results shown in (a).

The identification of *communicators* may provide a powerful tool for social dynamics. Here, we give an example by considering a merger between two systems. Suppose that two systems, P and Q, evolve according to the dynamics of Eq. (1) with the transition matrices M_P and M_Q which have the dimensions N_P and N_Q. We further assume that both P and Q are able to achieve some states of consensus, the *theorem* then implies that there exists integers n_p and n_q such that $M_P^{n_p}$ and $M_Q^{n_q}$ have a positive column locating respectively at, say, the αth and the βth. As the two systems are merged to form the system $R = P \cup Q$ by adding some connections between P and Q, the resultant transition matrix M_R takes the form of

$$M_R = \begin{pmatrix} M_P & C \\ C^\tau & M_Q \end{pmatrix}, \tag{17}$$

where the matrix C specifies the connections between P and Q, and C^τ is the transpose of C. Note that because of the added connections, some entries of M_P and M_Q may have to be modified; but, the positive entries remain to be positive after the modification, it does not affect the result obtained below. By defining $n_r = [n_p, n_q]$, the least common multiple of n_p and n_q, and setting $\widehat{M}_P = \left(M_P^{n_p}\right)^{n_r/n_p}$ and $\widehat{M}_Q = \left(M_Q^{n_q}\right)^{n_r/n_q}$, we then have

$$M_R^{n_r+1} \geq \begin{pmatrix} \widehat{M}_P & 0 \\ 0 & \widehat{M}_Q \end{pmatrix} \cdot M_R, \tag{18}$$

which yields

$$M_R^{n_r+1} \geq \begin{pmatrix} \widehat{M}_P \cdot M_P & \widehat{M}_P \cdot C \\ \widehat{M}_Q \cdot C^\tau & \widehat{M}_Q \cdot M_Q \end{pmatrix}. \tag{19}$$

Suppose that the new connection is added between the unit α of P and the unit β of Q, this gives a positive entry

(α, β) of C. Since the αth column of $\widehat{M}_P$ and the βth column of $\widehat{M}_Q$ are positive, we have the βth column of $\widehat{M}_P \cdot C$ and the αth column of $\widehat{M}_Q \cdot C^\tau$ being positive. Hence, the αth and the $(N_P+\beta)$th column of $M_R^{n_r+1}$ are positive, and a state of consensus for the merged system R can be achieved according to the *Theorem*. This gives the conclusion that only one connection between two *communicators* of different systems is required for the existence of a state of consensus in the merged system. However, the efficiency of reaching a consensus for the merged system depends on the levels of the connected *communicators* of different systems. To show the dependence explicitly, we consider the mergence of two systems defined in the Watts-Strogatz networks with $N = 100$, $k_0 = 4$ for $p = 0$ and 0.1. The previous model for the transition matrix with $s = 0.3$ is used to classify the *communicators* of two systems. All units are *the primary communicators* for the system $p = 0$, and the numbers of *communicators* at different levels are different for the system $p = 0.1$. By connecting a fixed unit of the system $p = 0$ to one of the *communicators* at a given level in the system $p = 0.1$, we use the enlarged transition matrix to calculate the time-steps of reaching the state of consensus from a strongly disorder state, and then calculate the average value over the time-steps required for different *communicators* at the same level in the system $p = 0.1$. The results are shown as the plot of the number of the average time-steps of reaching the state of consensus $c = 0.5$, denoted as $\langle T \rangle$, vs. the level of the connected *communicator* of the system $p = 0.1$, denoted as L, in Fig. 2. Our results indicate that the connection between a pair of *primary commutators* belonging to different systems provides the minimal and the most efficient way to have the merged system reaching the state of consensus.

In summary, we present a novel way for characterizing the process of reaching the state of consensus in a social system. The characterization provide not only the insights on the occurrence of system-wide harmonic behaviors but also a useful tool for the study of social dynamics. The foundation for the characterization is the *theorem* we establish, which can be viewed as an important extension of the Perron-Frobenius theorem.

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FIG. 1: The first appearance time of being a *communicator*, t_c (the upper part), and the corresponding degrees of connection, k (the lower part), for different units of the system, n. Here, the unit n is labelled in accordance with the order of t_c from small to large, and the system is defined on a Watts-Strogatz network with $N = 1000$, $k_0 = 4$, and $p = 0.1$.

FIG. 2: The average of the first appearance times, $\langle t_c \rangle$, of the *communicators* with the same degree of connection, k, as a function of k. The results are based on the data of Fig. 1.

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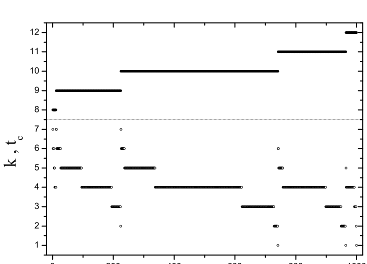
FIG. 3: The average time-steps of reaching the consensus for the merged system, $\langle T \rangle$, as a function of the level of the *communicators* of the second system, L, connected by a *primary communicator* of the first system. Two systems are defined on the Watts-Strogatz networks with $N = 100$, $k_0 = 4$, and the rewiring probability $p = 0$ for the first system and $p = 0.1$ for the second.



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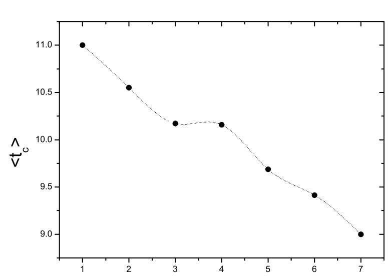

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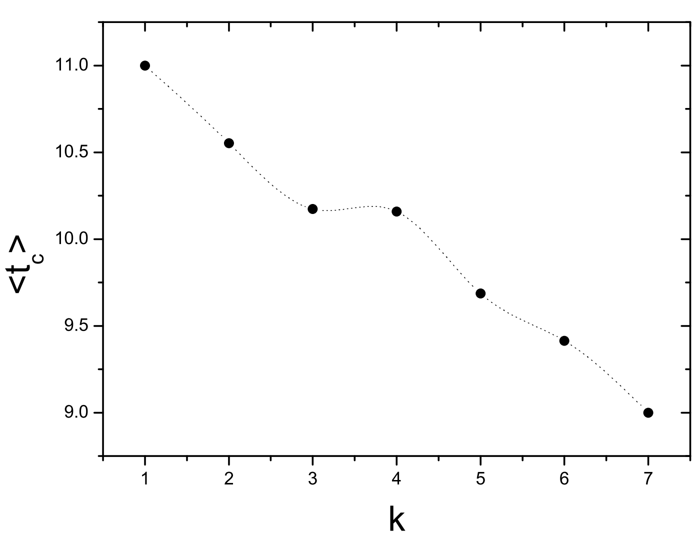

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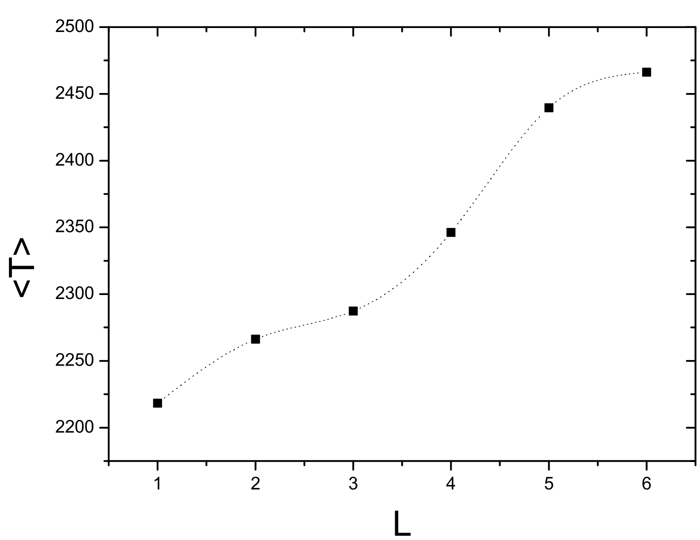
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